TEMPERATURE FIELD IN PLATES AND FLAT SHELLS WITH INTERNAL HEAT SOURCES

S. Ya. Yarema and M. P. Savruk

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Solutions of the problem of heat conduction with boundary conditions of the first, second, and third kinds are obtained for an infinite plate exposed to one of the following influences: instantaneous point heat source, initial temperature concentrated at a point, or instantaneous point action of a medium at its surface.

Green's functions of the problem of heat conduction for plates and flat shells. Let a homogeneous isotropic plate be heated by internal heat sources. The temperature at any point of the plate is given by the equation [1]

$$\frac{\partial \theta}{\partial \tau} - \Delta\theta = \psi_0(\alpha, \beta, \gamma, \tau) \tag{1}$$

and the boundary conditions

$$-a_1 \frac{\partial \theta}{\partial \gamma} + b_1\theta = \psi_1(\alpha, \beta, \tau) \quad \text{at} \quad \gamma = 0;$$

$$a_2 \frac{\partial \theta}{\partial \gamma} + b_2\theta = \psi_2(\alpha, \beta, \tau) \quad \text{at} \quad \gamma = 1;$$

$$\theta(\alpha, \beta, \gamma, \tau) = \psi_3(\alpha, \beta, \gamma) \quad \text{at} \quad \tau = 0. \tag{2}$$

Equation (1) is also applicable to thin-walled flat shells [2].

We will find the distribution of temperature θ^* in the plate when the functions ψ_i are given in the form

$$\psi_0(\alpha, \beta, \gamma, \tau) = \frac{\delta(\alpha-\alpha_0)}{\alpha_0}\,\delta(\beta-\beta')\,\delta(\gamma-\gamma_0)\,\delta(\tau);$$

$$\psi_1(\alpha, \beta, \tau) = \frac{\delta(\alpha-\alpha_1)}{\alpha_1}\,\delta(\beta-\beta')\,\delta(\tau);$$

$$\psi_2(\alpha, \beta, \tau) = \frac{\delta(\alpha-\alpha_2)}{\alpha_2}\,\delta(\beta-\beta')\,\delta(\tau);$$

$$\psi_3(\alpha, \beta, \gamma) = \frac{\delta(\alpha-\alpha_3)}{\alpha_3}\,\delta(\beta-\beta')\,\delta(\gamma-\gamma_3).$$

We represent $\delta(\beta - \beta')$ in series form:

$$\delta(\beta-\beta') = \frac{1}{\pi}\sum_{m=0}^{\infty} \varepsilon_m \cos m(\beta-\beta'),$$

and, correspondingly, write

$$\theta^* = \frac{1}{\pi}\sum_{m=0}^{\infty} \varepsilon_m \theta_m^* \cos m(\beta-\beta'). \tag{3}$$

The coefficients θ_m^* satisfy the equation

$$\frac{\partial \theta_m^*}{\partial \tau} - \frac{1}{\alpha}\frac{\partial}{\partial \alpha}\left(\alpha\frac{\partial \theta_m^*}{\partial \alpha}\right) + \frac{m^2}{\alpha^2} - \frac{\partial^2 \theta_m^*}{\partial \gamma^2} = \frac{\delta(\alpha-\alpha_0)}{\alpha_0}\,\delta(\gamma-\gamma_0)\,\delta(\tau) \tag{4}$$

and the boundary conditions

$$-a_1\frac{\partial \theta_m^*}{\partial \gamma} + b_1\theta_m^* = \frac{\delta(\alpha-\alpha_1)}{\alpha_1}\,\delta(\tau) \quad \text{at} \quad \gamma = 0;$$

$$a_2\frac{\partial \theta_m^*}{\partial \gamma} + b_2\theta_m^* = \frac{\delta(\alpha-\alpha_2)}{\alpha_2}\,\delta(\tau) \quad \text{at} \quad \gamma = 1;$$

$$\theta_m^*(\alpha, \gamma, \tau) = \frac{\delta(\alpha-\alpha_3)}{\alpha_3}\,\delta(\gamma-\gamma_3) \quad \text{at} \quad \tau = 0. \tag{5}$$

To solve Eq. (4) we use Doetsch integral transforms [3] with respect to the variable γ and Hankel transforms with respect to α. We denote the double transform of θ_m^* by Θ_{mn}, i.e.,

$$\Theta_{mn} = \int_0^\infty \alpha J_m(u\alpha)\,d\alpha \int_0^1 \theta_m^* Z_n(\gamma)\,d\gamma,$$

where

$$Z_n(\gamma) = A_n\cos\mu_n\gamma + B_n\sin\mu_n\gamma$$

are solutions of the problem

$$Z_n''(\gamma) + \mu_n^2 Z_n(\gamma) = 0 \tag{6}$$

with the following boundary conditions:

$$-a_1Z_n'(0) + b_1Z_n(0) = 0;$$

$$a_2Z_n'(1) + b_2Z_n(1) = 0. \tag{7}$$

The determinant of system (7), equated to zero, gives the characteristic equation for finding the eigenvalues μ_n^2:

$$\operatorname{tg}\mu_n = \mu_n(a_2b_1 + b_2a_1)/(a_1a_2\mu_n^2 - b_1b_2). \tag{8}$$

Investigation shows that the problem does not have zero eigenvalues $\mu_n^2 = 0$, except for the case of $b_1 = b_2 = 0$.

We find the coefficients A_n and B_n from (7) and also from the normalization condition

$$\int_0^1 Z_n^2(\gamma)\,d\gamma = 1.$$

After evaluation we obtain

$$B_n^2 = \frac{2b_1^2(a_2^2\mu_n^2 + b_2^2)}{(a_1^2\mu_n^2 + b_1^2)(a_2^2\mu_n^2 + b_2^2) + (a_1a_2\mu_n^2 + b_1b_2)(a_2b_1 + b_2a_1)};$$

$$A_n = \frac{a_1\mu_n}{b_1}B_n \quad \text{for} \quad n \geqslant 1;$$

$$B_0 = 0;$$

$$A_0 = \begin{cases} 1 & \text{at} \quad b_1 = b_2 = 0; \\ 0 & \text{otherwise}. \end{cases}$$

Multiplying expression (4) and the last of Eqs. (5) by the kernel of the transform and then integrating

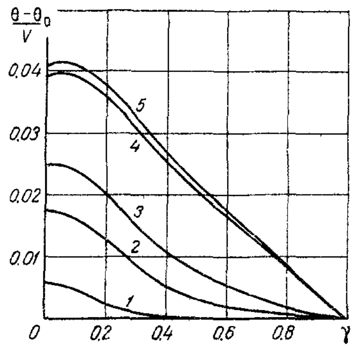

Fig. 1. Distribution of temperature field over thickness of plate: 1) at $\tau = 0.01$; 2) 0.05; 3) 0.1; 4) 0.5; 5) ∞.

within the corresponding limits, we have

$$\frac{d\,\Theta_{mn}}{d\,\tau} + (u^2 + \mu_n^2)\,\Theta_{mn} = \sum_{i=0}^{2} c_i(\gamma_i)\,Z_n(\gamma_i)\,J_m(u\,\alpha_i)\,\delta(\tau),$$

$$\Theta_{mn}(0) = J_m(u\,\alpha_3)\,Z_n(\gamma_3). \qquad (9)$$

Here, we have taken into account the fact that

$$\int_0^1 \frac{\partial^2\theta_m^*}{\partial\gamma^2}\,Z_n(\gamma)\,d\,\gamma = \sum_{i=1}^{2} c_i(\gamma_i)\,Z_n(\gamma_i)\,\frac{\delta(\alpha-\alpha_i)}{\alpha_i}\,\delta(\tau) - \mu_n^2\int_0^1 \theta_m^*\,Z_n(\gamma)\,d\,\gamma.$$

The solution of Eq. (9) will be

$$\Theta_{mn} = \sum_{i=0}^{3} c_i(\gamma_i) Z_n(\gamma_i)\,J_m(u\,\alpha_i)\exp\left[-(u^2+\mu_m^2)\,\tau\right],$$

where

$$c_i(\gamma_i) = \begin{cases} \dfrac{1}{a_i} & \text{at } \ a_i \neq 0; \\ (-1)^{i+1}\dfrac{d}{d\,\gamma_i} & \text{at } \ a_i = 0; \end{cases}$$

$$a_0 = a_3 = 1; \quad \gamma_1 = 0; \quad \gamma_2 = 1.$$

The inverse transformation gives

$$\theta_m^* = \sum_{i=0}^{3} c_i(\gamma_i)\,Z_n(\gamma_i)\,Z_n(\gamma)\exp(-\mu_n^2\tau)\times \int_0^\infty u J_m(u\,\alpha_i)\,J_m(u\,\alpha)\exp(-u^2\tau)\,du.$$

Using the formula for the addition of cylindrical functions, we sum θ_m^*, in accordance with (3), and after integration obtain

$$\theta^* = \sum_{i=0}^{3} G_i(\alpha,\ \alpha_i,\ \beta,\ \beta',\ \gamma,\ \gamma_i,\ \tau), \qquad (10)$$

where

$$G_i(\alpha,\ \alpha_i,\ \beta,\ \beta',\ \gamma,\ \gamma_i,\ \tau) = \frac{1}{4\pi\tau}\exp\left(-\frac{R_i^2}{4\tau}\right)\times \sum_{n=0}^{\infty} c_i(\gamma_i)\,Z_n(\gamma)\,Z_n(\gamma_i)\exp(-\mu_n^2\tau);$$

$$R_i = \sqrt{\alpha^2 + \alpha_i^2 - 2\alpha\alpha_i\cos(\beta-\beta')}\,. \qquad (11)$$

Results (11) can be used in an arbitrary coordinate system on the surface of the plate ($\gamma = 0$), if it is kept in mind that R is the distance between the center of action and the instantaneous point. In particular, in a Cartesian coordinate system $R_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2}$.

Each individual solution G_i gives the temperature field in the plate due to just one factor: concentrated instantaneous heat source ($i = 0$), concentrated instantaneous action of the medium at the surfaces of the plate ($i = 1, 2$), concentrated action of the initial temperature ($i = 3$).

Solutions (11) may be regarded as Green's functions of the problem (1), (2), which can be used to solve that problem for an arbitrary distribution of heat source intensity and arbitrary boundary conditions given by the functions ψ_i:

$$\theta(\alpha,\ \beta,\ \gamma,\ \tau) = \int_0^\infty \alpha_0\,d\,\alpha_0 \int_0^{2\pi} d\,\beta' \int_0^1 d\,\gamma_0 \times \int_0^\tau G_0(\alpha,\ \alpha_0,\ \beta,\ \beta',\ \gamma,\ \gamma_0,\ \tau')\times\psi_0(\alpha_0,\ \beta',\ \tau-\tau')\,d\,\tau' +$$

$$+\sum_{i=1}^{2}\int_0^\infty \alpha_i\,d\,\alpha_i \int_0^{2\pi} d\,\beta' \int_0^\tau G_i(\alpha,\ \alpha_i,\ \beta,\ \beta',\ \gamma,\ \gamma_i,\ \tau')\times \psi_i(\alpha_i,\ \beta',\ \tau-\tau')\,d\,\tau' + \int_0^\infty \alpha_3\,d\,\alpha_3 \int_0^{2\pi} d\,\beta' \times$$

$$\times \int_0^1 G_3(\alpha,\ \alpha_3,\ \beta,\ \beta',\ \gamma,\ \gamma_3,\ \tau)\times\psi_3(\alpha_3,\ \beta',\ \gamma_3)\,d\,\gamma_3. \qquad (12)$$

We note that result (12) can be obtained mathematically from Eq. (1) for homogeneous boundary conditions (2) if the heat source intensity is given by a specific function, i.e.,

$$\frac{\partial\theta}{\partial\tau} - \Delta\theta = \psi_0(\alpha,\ \beta,\ \gamma,\ \tau) + \sum_{i=1}^{2} c_i(\gamma)\,\psi_i(\alpha,\ \beta,\ \tau)\,\delta(\gamma-\gamma_i) + \delta(\tau)\,\psi_3(\alpha,\ \beta,\ \gamma);$$

$$-a_1\frac{\partial\theta}{\partial\gamma} + b_1\theta = 0 \quad \text{at} \quad \gamma = 0;$$

$$a_2\frac{\partial\theta}{\partial\gamma} + b_2\theta = 0 \quad \text{at} \quad \gamma = 1;$$

$$\theta(\alpha,\ \beta,\ \gamma,\ \tau) = 0 \quad \text{at} \quad \tau = 0.$$

Consequently, problem (1), (2) reduces to the problem with homogeneous boundary conditions. This result can be formulated as follows: the action of the medium on the plate is equivalent to heat sources (if $a_i \neq 0$) or dipoles (if $a_i = 0$) distributed over the surfaces of the plate, and the action of the initial temperature is equivalent to instantaneous heat sources acting at the initial instant inside the plate. This conclusion was reached in [4] in the case of the one-dimensional problem with boundary conditions of the first kind ($a_i = 0$).

Our method employing Doetsch integral transforms gives the solution in the form of series in eigenfunctions, these series converging absolutely and uniformly in all closed regions on the interval (0, 1).

The constants a_1, a_2, b_1, b_2 in boundary conditions (2) may take arbitrary nonnegative values, i.e., expression (12) may give a solution of Eq. (1) for boundary conditions of the first, second, or third kind on each of the boundary surfaces. If we take i = 0 in (11) and consider the cases $a_1 = a_2 = 0$, $b_1 = b_2 = 0$ and $a_1 = a_2$, $b_1 = b_2$, we obtain the known [5] solutions for instantaneous point heat sources.

Integrating (11) with respect to β from 0 to 2π, we obtain the Green's function of the axisymmetric problem

$$G_i(\alpha, \alpha_i, \tau) = \frac{1}{2\tau} \exp\left(-\frac{\alpha^2 + \alpha_i^2}{4\tau}\right) \times$$

$$\times I_0\left(\frac{\alpha\alpha_i}{2\tau}\right) \sum_{n=0}^{\infty} c_i(\gamma_i) Z_n(\gamma) Z_n(\gamma_i) \exp(-\mu_n^2 \tau).$$

In the particular case of i = 1 and $a_1 = a_2 = 1$, $b_1 = b_2 = 0$, this formula coincides with that given in [9].

Axisymmetric problem. Normal distribution of heat source intensity. We will consider the problem of determining the temperature field in a plate that is heated by heat sources with an intensity given with respect to the α coordinate by a Gaussian distribution law, i. e.,

$$\psi_0(\alpha, \gamma) = \varphi(\gamma) \exp(-\alpha^2/4k^2). \tag{13}$$

We assume that the form of the boundary conditions is arbitrary, but for simplicity consider that the functions ψ_1 and ψ_2, characterizing the action of the medium on the surfaces of the plate, are constant and that before the process begins the corresponding stationary regime has been established, i. e.,

$$\psi_1 = \text{const}, \quad \psi_2 = \text{const},$$

$$\psi_3(\alpha, \beta, \gamma) = \theta_0(\gamma) =$$

$$= \frac{\psi_1 [b_2(1-\gamma) + a_2] + \psi_2 (b_1 \gamma + a_1)}{b_1 b_2 + a_2 b_1 + b_2 a_1}.$$

Now, using Eq. (12), we obtain

$$\theta - \theta_0(\gamma) = k^2 \sum_{n=0}^{\infty} Z_n(\gamma) \overline{\varphi(\mu_n)} \exp(\mu_n^2 k^2) \times$$

$$\times \left\{ J_1\left(\frac{\alpha}{2k}, \mu_n k\right) - J_1\left(\frac{\alpha}{2\sqrt{k^2+\tau}}, \mu_n \sqrt{k^2+\tau}\right)\right\}, \tag{14}$$

where

$$\overline{\varphi(\mu_n)} = \int_0^1 \varphi(\gamma_0) Z_n(\gamma_0)\, d\gamma_0.$$

Here, $J_n(x, y)$ denotes the integral

$$J_n(x, y) = \int_1^{\infty} \exp\left(-\frac{x^2}{u} - y^2 u\right) \frac{\partial u}{u^n}.$$

Expanding $\exp(-x^2/u)$ in the integrand in series, we can represent the function $J_n(x, y)$ in the form

$$J_n(x, y) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{m!} E_{m+n}(y^2). \tag{15}$$

The functions $E_{m+n}(y^2) = \int_1^{\infty} \exp(-y^2 u) \frac{du}{u^{n+m}}$ were tabulated in [6].

At large values of x, when series (15) converges slowly, by using the asymptotic expansion of $E_{m+n}(y^2)$

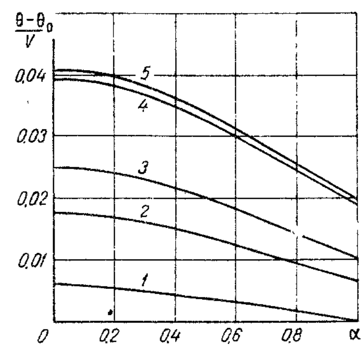

Fig. 2. Temperature distribution at surface of plate: 1–5) see Fig. 1.

[6] we can obtain the following formulas for computing $J_n(x, y)$ (n = 0,1):

$$J_0(x, y) = 2\,\frac{x}{y}\,K_1(2xy) - \frac{\exp(-x^2-y^2)}{x^2} \times$$

$$\times \left[1 + \frac{y^2-2}{x^2} + \frac{y^4 - 6y^2 + 6}{x^4} + O(x^{-6})\right];$$

$$J_1(x, y) = 2K_0(2xy) - \frac{\exp(-x^2-y^2)}{x^2} \times$$

$$\times \left[1 + \frac{y^2-1}{x^2} + \frac{y^4 - 4y^2 + 2}{x^4} + O(x^{-6})\right].$$

We have calculated the temperature field in a plate from Eq. (14) for the case when at one surface ($\gamma = 0$) we are given heat exchange with a medium at temperature θ_0 characterized by the Biot number $b_1 = \varepsilon l/\lambda = 1$, and at the other ($\gamma = 1$) the temperature θ_0. Consequently, $b_1 = b_2 = a_1 = 1$, $a_2 = 0$, $\psi_1 = \psi_2 = \psi_3 = \theta_0$. The law of variation of heat source intensity over the thickness of the plate is exponential, i. e., $\varphi(\gamma) = V\exp(-7\gamma)$, and the coefficient k in (13) is taken equal to 0.5.

Figure 1 shows the distribution of the temperature field over the thickness of the plate at the center of heating ($\alpha = 0$), and Fig. 2 the temperature distribution at the surface of the plate ($\gamma = 0$). We note that the variation of temperature along the radius of the plate is much smoother than the distribution of heat source intensity.

The problem considered corresponds roughly to the action of a beam of penetrating radiation (particle flux) normal to the surface of the plate, when the radiation intensity in the beam varies according to a normal law [7] and the heat release in the interior of the plate is determined by the function $\varphi(\gamma)$.

Uniform circular distribution of heat source intensity. When the beam of radiation is strongly concentrated it is more rational to consider that the intensity of the sources is uniformly distributed over a circle of radius a [9]:

$$\psi_0(\alpha, \gamma) = \begin{cases} \varphi(\gamma) & \text{at } \alpha < a, \\ 0 & \text{at } \alpha > a. \end{cases} \tag{16}$$

Assuming that the boundary conditions remain as before, and applying formula (12), we obtain

$$\theta(\alpha, \gamma, \tau) - \theta_0(\gamma) = \tag{17}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} Z_n(\gamma) \overline{\varphi(\mu_n)} \times$$

$$\times \int_0^{\tau} P(\alpha, \tau', a) \exp\left(-\frac{\alpha^2}{4\tau'} - \mu_n^2 \tau'\right) \frac{d\tau'}{\tau'}, \quad (17) \text{ (cont'd)}$$

where the integral

$$P(\alpha, \tau', a) = \int_0^a \alpha_0 \exp\left(-\frac{\alpha_0^2}{4\tau'}\right) I_0\left(\frac{\alpha\alpha_0}{2\tau'}\right) d\alpha_0$$

is called the P function. A table of values of this function is given in [8].

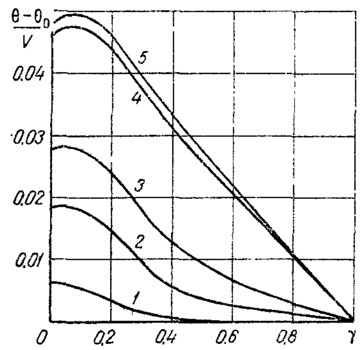

Fig. 3. Calculation of temperature field of plate: 1–5) see Fig. 1.

As $\tau \to \infty$, expression (17) can be simplified:

$$\theta(\alpha, \gamma, \infty) - \theta_0(\gamma) = \sum_{n=0}^{\infty} Z_n(\gamma) \overline{\varphi(\mu_n)} \times$$

$$\times \left\{\left[\frac{1}{\mu_n^2} - \frac{a}{\mu_n} K_1(a\mu_n) I_0(\alpha\mu_n)\right] \eta(a-\alpha) + \right.$$

$$\left. + \frac{a}{\mu_n} I_1(a\mu_n) K_0(\alpha\mu_n) \eta(\alpha - a)\right\};$$

at $\alpha = 0$, when $P(0, \tau', a) = 2\tau'[1 - \exp(-a^2/4\tau')]$,

$$\theta(0, \gamma, \tau) - \theta_0(\gamma) =$$

$$= \sum_{n=0}^{\infty} Z_n(\gamma) \overline{\varphi(\mu_n)} \left[\frac{1 - \exp(-\mu_n^2 \tau)}{\mu_n^2} - \frac{a}{\mu_n} K_1(a\mu_n) + \right.$$

$$\left. + \tau J_0\left(\frac{a}{2\sqrt{\tau}}, \mu_n\sqrt{\tau}\right)\right]. \quad (18)$$

Equation (18) was used to calculate the temperature field of a plate. The numerical data were taken from the previous problem and the radius a was taken equal to 2k, starting from the condition that the amount of heat released in the plate is the same for distributions (13) and (16). The results of the calculation are presented in Fig. 3.

As may be seen from the graphs, the temperature at the center of heating for a uniform circular distribution of heat source intensity is somewhat greater than for a normal distribution. The shape of the temperature distribution curves is roughly the same in both cases: they have a more or less clearly expressed maximum whose height and location vary with τ.

We note that similar problems for nonpenetrating radiation (i. e., in the absence of internal heat sources) were investigated by Sticker [9].

NOTATION

θ is the temperature, θ_0 is initial temperature; r, φ, and z are the cylindrical coordinates; $\alpha = r/l$, $\beta = \varphi$, $\gamma = z/l$ are the dimensionless coordinates; l is the thickness of plate; $\tau = \varkappa t/l^2$ is the Fourier number; t is time; $\varkappa$ is thermal diffusivity; λ is thermal conductivity; ε is the heat transfer coefficient; $\psi_0(\alpha,\beta,\gamma,\tau) = (l^2/\lambda)W(\alpha,\beta,\gamma,\tau)$; $V = (l^2/\lambda)W$; $W(\alpha,\beta,\gamma,\tau)$ is heat source intensity; $E_n(y)$ is the integroexponential function; $\delta(x - x_1)$ is the Dirac function; Δ is the Laplace operator in coordinates α,β,γ; $J_n(x)$ is the Bessel function; $I_n(x)$ and $K_n(x)$ are modified Bessel functions; $\varepsilon_m = \begin{cases} 0.5, & m = 0 \\ 1, & m \geq 1, \end{cases}$ $\eta(\alpha - a) = \begin{cases} 1, & \alpha > a \\ 0, & \alpha < a. \end{cases}$

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Physicomechanics Institute
AS UkrSSR, L'vov